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A NOTE ON THE PROBLEM OF FLEXURE OF PRISMATICAL BEAMS

E. REISSNER
Department of Applied Mechanics and Engineering Sciences, University of California, San Diego, La Jolla, CA 92093, U.S.A.

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Abstract—In the classical St Venant flexure solution all differential equations and certain boundary conditions are satisfied exactly, while other boundary conditions are satisfied approximately only.

In this note an alternate solution to the flexure problem is considered, in which the components of stress which do *not* act over the end sections of the beam are again set identically equal to zero, the same as in the St Venant solution. However, in place of St Venant's exact satisfaction of the stress–strain differential equations we now associate the approximate satisfaction of boundary conditions with an approximate satisfaction of the stress–strain relations, via use of the principle of minimum complementary energy.

The possible applications of the foregoing results are illustrated by the solution of two problems concerning the effect of transverse shear deformability and of end section warping constraint on deflection and shear center location for a circular and a semi-circular cross-section cantilever, respectively.

INTRODUCTION

In what follows the problem of flexure is understood to be the problem of a prismatical beam which is fixed at one end and transversely displaced at the other end, within the framework of the linear theory of elasticity.

The classical approximate solution of this problem by St Venant consists of an exact satisfaction of the differential equations, and of the traction conditions over the cylindrical boundary portion, by way of a "semi-inverse" procedure in which all components of stress which do not act over the plane end-surfaces are assumed to vanish identically. With this the only conditions on the solution of the problem which are not satisfied exactly are the boundary conditions for the two end sections of the beam.

Among the specific problems for which use of the St Venant flexure solution is associated with ambiguous results are the problem of determining *transverse shear corrections* for deflections in accordance with the "elementary" theory of beams, and the problem of determining the coordinates of a point in the cross-section which is appropriately designated as the *center of shear* of the cross-section.

We are concerned in this note with extensions and simplification of earlier considerations of the two indicated specific problems in Nair and Reissner (1975), Reissner (1979, 1983) and Reissner and Tsai (1972), based on an idea that in conjunction with the use of St Venant's semi-inverse stress assumptions there are advantages in associating the non-exact satisfaction of end-section displacement boundary conditions with a suitable non-exact satisfaction of the differential equations involving components of displacement.

THE BOUNDARY VALUE PROBLEM

We consider a body bounded by a cylindrical surface $f(x,y) = 0$ and by planes $z = 0$ and $z = L$. We stipulate that the differential equations of the problem are the three homogeneous equilibrium equations

$$\sigma_{x,x} + \tau_{xy,y} + \tau_{xz,z} = 0, \ldots, \quad \tau_{xz,x} + \tau_{yz,y} + \sigma_{z,z} = 0 \tag{1}$$

in conjunction with six stress displacement equations

$$u_{,x} = \frac{\partial W}{\partial \sigma_x}, \quad u_{,y} + v_{,x} = \frac{\partial W}{\partial \tau_{xy}}, \ldots, \quad w_{,z} = \frac{\partial W}{\partial \sigma_z} \tag{2}$$

in terms of a complementary energy function which here is assumed to be homogeneous of the second degree in $\sigma_x, \tau_{xy}, \ldots, \sigma_z$.

The boundary conditions for the system (1) and (2) consist of three conditions of absent tractions over the surface $f = 0$, together with three conditions of absent displacements over the surface $z = 0$, and of three conditions of loading in the mixed form

$$z = L; \quad u = U, \quad v = V, \quad \sigma_z = 0 \tag{3}$$

with U and V as given constants.

The problem as stated is equivalent to the Minimum Complementary Energy variational equation

$$\delta \int\int\left[\int W\,\mathrm{d}z - (U\tau_{xz} + V\tau_{yz})_{z=L}\right]\mathrm{d}x\,\mathrm{d}y = 0 \tag{4}$$

with the equilibrium differential equations and the traction boundary conditions as constraints, and with the stress displacement equations and displacement boundary conditions as Euler equations.

In connection with this problem we are here interested, in particular, in the values of the two transverse force components

$$P = \int \tau_{xz}\,\mathrm{d}a, \quad Q = \int \tau_{yz}\,\mathrm{d}a \tag{5}$$

and in the value of the axial moment component, or torque

$$T = \int (x\tau_{yz} - y\tau_{xz})\,\mathrm{d}a, \tag{6}$$

where here and in what follows $\int \mathrm{d}a \equiv \int\int \mathrm{d}x\,\mathrm{d}y$.

A RAYLEIGH–RITZ TYPE SOLUTION BASED ON THE ST VENANT STRESS ASSUMPTIONS

We begin with the St Venant assumptions

$$\sigma_x = 0, \quad \tau_{xy} = 0, \quad \sigma_y = 0 \tag{7}$$

for his solution of the problem of flexure and use the constraint conditions in (1) and (3) in order to obtain as expressions for the remaining components of stress

$$\tau_{xz} = \tau_x(x, y), \quad \tau_{yz} = \tau_y(x, y), \quad \sigma_z = (L - z)(\tau_{x,x} + \tau_{y,y}). \tag{8}$$

The expression for W is, in the present context,

$$W = \frac{\tau_x^2}{2G_x} + \frac{\tau_y^2}{2G_y} + \frac{\tau_x\tau_y}{G_{xy}} + \frac{\sigma_z^2}{2E} + \frac{\sigma_z\tau_x}{E_x} + \frac{\sigma_z\tau_y}{E_y}, \tag{9}$$

where we here stipulate that the six denominators are given functions of x and y with the restriction that W be positive definite.

With (8) and (9) the St Venant *solution* of the flexure problem is based on an exact satisfaction of (2), subject to the special case assumption

$$W=\frac{\tau_x^2+\tau_y^2}{2G}+\frac{\sigma_z^2}{2E}, \tag{10}$$

with G and E being independent of x and y. As a consequence of these assumptions it then follows that σ_z comes out in the form $(A_0+A_1x+A_2y)(L-z)$, with suitable constants A_i, and that it is *not* possible to satisfy the prescribed displacement boundary conditions exactly.

In what follows we do not attempt an exact satisfaction of (2) but rather associate the approximate satisfaction of displacement boundary conditions with an approximate satisfaction of (2), through use of the variational equation (4) in conjunction with eqns (8) and (9).

The introduction of (8) and (9) into (4) leaves, upon carrying out the integration with respect to z, the two-dimensional variational equation

$$\delta\int\left[\frac{\tau_x^2}{2G_x}+\frac{\tau_y^2}{2G_y}+\frac{\tau_x\tau_y}{G_{xy}}+\frac{L^2}{3}\frac{(\tau_{x,x}+\tau_{y,y})^2}{2E}+\frac{L}{2}(\tau_{x,x}+\tau_{y,y})\left(\frac{\tau_x}{E_x}+\frac{\tau_y}{E_y}\right)-\frac{U}{L}\tau_x-\frac{V}{L}\tau_y\right]\mathrm{d}a=0, \tag{11}$$

with the boundary condition

$$f=0;\quad \tau_x\,\mathrm{d}y-\tau_y\,\mathrm{d}x=0 \tag{12}$$

as the only constraint condition.

The Euler equations for (11) come out in the form

$$\frac{\tau_x}{G_x}+\frac{\tau_y}{G_{xy}}-\frac{L^2}{3}\left(\frac{\tau_{x,x}+\tau_{y,y}}{E}\right)_{,x}+\frac{L}{2}\frac{\tau_{x,x}+\tau_{y,y}}{E_x}-\frac{L}{2}\left(\frac{\tau_x}{E_x}+\frac{\tau_y}{E_y}\right)_{,x}=\frac{U}{L}, \tag{13a}$$

$$\frac{\tau_x}{G_{xy}}+\frac{\tau_y}{G_y}-\frac{L^2}{3}\left(\frac{\tau_{x,x}+\tau_{y,y}}{E}\right)_{,y}+\frac{L}{2}\frac{\tau_{x,x}+\tau_{y,y}}{E_y}-\frac{L}{2}\left(\frac{\tau_x}{E_x}+\frac{\tau_y}{E_y}\right)_{,y}=\frac{V}{L}. \tag{13b}$$

We do not, in this account, concern ourselves with the solution of these equations in their generality, but will instead obtain a reduction of the problem to a single second order differential equation for the case of orthotropy.

REDUCTION TO ONE SECOND ORDER DIFFERENTIAL EQUATION FOR ORTHOTROPIC BEAMS

Setting $G_{xy}=E_x=E_y=\infty$ the system (13a, b) reduces to

$$\frac{\tau_x}{G_x}-\frac{L^2}{3}\left(\frac{\tau_{x,x}+\tau_{y,y}}{E}\right)_{,x}=\frac{U}{L},\quad \frac{\tau_y}{G_y}-\frac{L^2}{3}\left(\frac{\tau_{x,x}+\tau_{y,y}}{E}\right)_{,y}=\frac{V}{L}. \tag{14}$$

Upon the appropriate differentiation with respect to x and y we deduce from this the relation

$$\left(\frac{\tau_x}{G_x}\right)_{,y}-\left(\frac{\tau_y}{G_y}\right)_{,x}=0, \tag{15}$$

which implies as expressions for τ_x and τ_y in terms of a function ϕ;

$$\tau_x = G_x\phi_{,x}, \quad \tau_y = G_y\phi_{,y}. \tag{16}$$

The introduction of (16) into (14) and subsequent integrations with respect to x and y leave as one differential equation for ϕ

$$\phi - \frac{L^2}{3}\frac{(G_x\phi_{,x})_{,x} + (G_y\phi_{,y})_{,y}}{E} = \frac{U}{L}x + \frac{V}{L}y + \frac{C}{L}. \tag{17}$$

In view of the fact that, necessarily,

$$\int \sigma_z \, \mathrm{d}a = (L-z)\int (\tau_{x,x} + \tau_{y,y})\,\mathrm{d}a = 0 \tag{18}$$

we deduce from (16) and (17) as a relation for C

$$L\int \phi E \,\mathrm{d}a = U\int xE\,\mathrm{d}a + V\int yE\,\mathrm{d}a + C\int E\,\mathrm{d}a \tag{19}$$

and the boundary condition for ϕ follows from (16) and (12) in the form

$$f = 0; \quad G_x\phi_{,x}\,\mathrm{d}y - G_y\phi_{,y}\,\mathrm{d}x = 0. \tag{20}$$

The force and torque expressions in (5) and (6) become

$$P = \int G_x\phi_{,x}\,\mathrm{d}a, \quad Q = \int G_y\phi_{,y}\,\mathrm{d}a, \tag{21a}$$

$$T = \int (xG_y\phi_{,y} - yG_x\phi_{,x})\,\mathrm{d}a. \tag{21b}$$

With reference to the solution of (17) it is of importance to observe that with representative constant moduli G_0 and E_0 and with b as a representative linear dimension in the cross-section it is possible to expand the function ϕ in powers of a parameter $(E_0/G_0)(b^2/L^2)$ with the leading term of this expansion being determined by (17) without the ϕ-term on the left. It turns out that retaining this ϕ-term is required in connection with the determination of transverse shear deformation effects for the results of the "elementary" theory of bending, as well as for the results concerning the location of a center of shear.

As far as the evaluation of the integrals in (21) is concerned we have that as a consequence of the relation

$$\int (x, y)\sigma_z\,\mathrm{d}a = (P, Q)(z-L) \tag{22}$$

it is possible to transform (21a) into

$$(P, Q) = 3L^{-3}\int (x, y)(Ux + Vy + C - L\phi)E\,\mathrm{d}a. \tag{23}$$

While there appears to be no analogous transformation for (21b) it is possible to deduce from (21b) a generalization of a formula for T in terms of the warping function χ of St Venant *torsion* as determined for the present case with variable G_x and G_y by the boundary value problem:

$$(G_x\chi_{,x})_{,x}+(G_y\chi_{,y})_{,y} = yG_{x,x}-xG_{y,y}, \tag{24}$$

$$f = 0; \quad G_x\chi_{,x}\,\mathrm{d}y-G_y\chi_{,y}\,\mathrm{d}x = yG_x\,\mathrm{d}y-xG_y\,\mathrm{d}x. \tag{25}$$

A use of Green's theorem, in the form

$$\int\{[(G_x\chi_{,x})_{,x}+(G_y\chi_{,y})_{,y}]\phi-[(G_x\phi_{,x})_{,x}+(G_y\phi_{,y})_{,y}]\chi\}\,\mathrm{d}a$$
$$= \oint[\phi(G_x\chi_{,x}\,\mathrm{d}y-G_y\chi_{,y}\,\mathrm{d}x)-\chi(G_x\phi_{,x}\,\mathrm{d}y-G_y\phi_{,y}\,\mathrm{d}x)] \tag{26}$$

in conjunction with the use of (17), (20), (24) and (25) makes it possible to transform (21b) into

$$T = 3L^{-3}\int(Ux+Vy+C-L\phi)\chi E\,\mathrm{d}a \tag{27}$$

with this formula being equivalent to the result in Trefftz (1935) upon omitting the additive term $L\phi$.

We finally note that in place of the variational equation (11) with $G_{xy} = E_x = E_y = \infty$ and with the constraint condition (12) it is evidently possible to state a variational equation for ϕ, without constraint boundary conditions, in the form

$$\delta\int[G_x\phi_{,x}^2+G_y\phi_{,y}^2+3EL^{-2}\phi^2-6EL^{-3}(Ux+Vy+C)\phi]\,\mathrm{d}a = 0. \tag{28}$$

It remains to establish the possible advantages of (28) in comparison with (11) in connection with their use for the purpose of finite element calculations.

THE SHEAR CENTER PROBLEM

Given that the torque T as in (6) may be considered to be caused by forces P and Q with lines of action y_P and x_Q we ask for the values of x_Q and y_P such that the end cross-section translates, without rotating, in accordance with (3). We designate the point of intersection of these distinguished lines of action as the Center of Shear and write x_S, y_S for the coordinates of this point, in place of x_Q, y_P. We have then as the relation defining x_S, y_S,

$$T = x_SQ-y_SP. \tag{29}$$

From the linearity of the boundary value problem (1)–(3) and the defining relations (5) and (6) it follows that the expressions for P, Q, T in terms of U, V will be of the form

$$(P,Q,T) = (K_{QU},K_{QU},K_{TU})U+(K_{PV},K_{QV},K_{TV})V \tag{30}$$

with the values of the stiffness coefficients K depending on the solutions of the boundary value problem (1)–(3), with $V = 0$ or $U = 0$, respectively.

The introduction of (30) into (29) leads to the relation

$$(K_{TU}-x_SK_{PU}+y_SK_{QU})U+(K_{TV}-x_SK_{PV}+y_SK_{QV})V = 0. \tag{31}$$

Inasmuch as (31) has to hold *identically* in U and V there follows as a system of two simultaneous equations for x_S and y_S

$$K_{PU}x_S - K_{QU}y_S = K_{TU}, \quad K_{PV}x_S - K_{QV}y_S = K_{TV}. \tag{32}$$

Remark

The results in accordance with (32) are equivalent to results which have been obtained in Reissner and Tsai (1972) and Reissner (1979) on the basis of a more general boundary value problem with $u = U - y\Theta$, $v = V + x\Theta$ in (3). For this more general problem it is possible to obtain expressions for x_S and y_S in terms of flexibility coefficients rather than stiffness coefficients which are of a simpler appearance than the corresponding stiffness coefficient relations.

POLAR COORDINATE SOLUTIONS FOR UNIFORM ORTHOTROPIC BEAMS

With $G_x = G_y = G$ and $G = \text{const.}$ and with $x = r\cos\theta$, $y = r\sin\theta$ and $G\phi = \psi$ we may write in place of (16)

$$\tau_r = \psi_{,r}, \quad \tau_\theta = r^{-1}\psi_{,\theta}, \tag{33}$$

and in place of (17)

$$\nabla^2\psi - \frac{3E}{GL^2}\psi = -\frac{3E}{L^3}(Ur\cos\theta + Vr\sin\theta + C), \tag{34}$$

where $\nabla^2 = (\ )_{,rr} + r^{-1}(\ )_{,r} + r^{-2}(\ )_{,\theta\theta}$. In association with this we write (19) and (23) in the form

$$G\iint(Ur\cos\theta + Vr\sin\theta + C)r\,\mathrm{d}r\,\mathrm{d}\theta = L\iint\psi r\,\mathrm{d}r\,\mathrm{d}\theta \tag{35}$$

and

$$\begin{pmatrix}P\\Q\end{pmatrix} = \iint\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}\left(Ur\cos\theta + Vr\sin\theta + C - \frac{L}{G}\psi\right)Er^2\,\mathrm{d}r\,\mathrm{d}\theta. \tag{36}$$

Furthermore, we write in place of (21b)

$$T = \iint r\tau_\theta r\,\mathrm{d}r\,\mathrm{d}\theta = \iint\psi_{,\theta}r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{37}$$

In restating the boundary condition (20) we will limit ourselves to the case of a ring sector cross-section $r_i \leqslant r \leqslant r_0$, $-\alpha \leqslant \theta \leqslant \alpha$ for which this condition takes on the form

$$\psi_{,r}(r_i,\theta) = \psi_{,r}(r_0,\theta) = \psi_{,\theta}(r,-\alpha) = \psi_{,\theta}(r,\alpha) = 0. \tag{38}$$

The third and fourth relations in (38) indicate that solutions of (34) can be taken as

$$\psi = \sum_{n=0}^{\infty} f_n(r)\sin\frac{2n+1}{2}\frac{\pi}{\alpha}\theta + g_n(r)\cos\frac{n\pi}{\alpha}\theta. \tag{39}$$

It is evident that upon introducing on the right of (34) the expansions

$$\sin\theta = \sum a_n\sin\frac{2n+1}{2}\frac{\pi}{\alpha}\theta, \quad \cos\theta = \sum b_n\cos\frac{n\pi}{\alpha}\theta \tag{40}$$

and upon assuming that $E = E(r)$ the functions f_n and g_n will be solutions of uncoupled ordinary second order differential equations. Cases of particular simplicity are given when

$$U = C = 0, \quad \alpha = \pi/2; \quad a_0 = 1, \quad a_n = 0, \quad n = 1, 2, \ldots, \tag{41a}$$

$$V = 0, \quad \alpha = \pi; \quad b_1 = 1, \quad b_n = 0, \quad n = 0, 2, 3, \ldots \tag{41b}$$

THE SHEAR CENTER PROBLEM FOR A SEMI-CIRCULAR CROSS-SECTION

We have for this problem $U = C = 0$, and then from (35) and (39) that $\psi = f_0(x) \sin\theta$ where

$$f_0'' + \frac{1}{r} f_0' - \left(\frac{1}{r^2} + \frac{3E}{GL^2}\right) f_0 = -\frac{3EV}{L^3} r. \tag{42}$$

If we assume from now on that $E =$ const. then the solution of (42) is, in terms of modified Bessel functions,

$$f_0 = c_1 I_1\left(\beta \frac{r}{r_0}\right) + c_2 K_1\left(\beta \frac{r}{r_0}\right) + \frac{GV}{L} r, \quad \beta^2 = \frac{3E}{G}\left(\frac{r_0}{L}\right)^2, \tag{43}$$

with the constants c_1, c_2 following from the remaining conditions in (38).

A further limitation to the case $r_i = 0$ makes $c_2 = 0$ and

$$\psi = GV \frac{r_0}{L}\left(\frac{r}{r_0} - \frac{I_1(\beta r/r_0)}{\beta I_1'(\beta)}\right) \sin\theta. \tag{44}$$

The introduction of (44) into (36) and (37) gives

$$Q = 3\pi EV \frac{r_0^4}{L^3} \int_0^1 \left(\rho^3 - \frac{I_1(\beta\rho)\rho^2}{2\beta I'(\beta)}\right) \mathrm{d}\rho \tag{45}$$

and

$$T = 2GV \frac{r_0^3}{L} \int_0^1 \left(\rho^2 - \frac{I_1(\beta\rho)\rho}{\beta I_1'(\beta)}\right) \mathrm{d}\rho, \tag{46}$$

where $\rho = r/r_0$. With this we obtain on the basis of (29) as an expression for the shear center coordinate, with $y_S = 0$:

$$\frac{x_S}{x_0} = \frac{T}{Qx_0} = \frac{2}{\pi\beta^2} \frac{\int_0^1 [I_1'(\beta) - I_1(\beta\rho)/\beta\rho]\rho^2 \,\mathrm{d}\rho}{\int_0^1 [2I_1'(\beta) - I_1(\beta\rho)/\beta\rho]\rho^3 \,\mathrm{d}\rho}. \tag{47}$$

Given the form of the expression for β we expect that (47) should reduce, in the limit of vanishing β, to the known result $(x_S/x_0)_{r_0/L=0} = 8/5\pi$. An evaluation of (47) in the range of relatively small values of β leads, with the help of the expansion

$$I_1(\beta) = \tfrac{1}{2}\beta + \tfrac{1}{16}\beta^3 + \tfrac{1}{384}\beta^5 + \cdots \tag{48}$$

to the following approximate result:

$$\frac{x_S}{x_0} \approx \frac{8}{5\pi}\,\frac{1+5\beta^2/63}{1+2\beta^2/3} \approx \frac{8}{5\pi}\left(1-\frac{37}{21}\,\frac{E}{G}\,\frac{r_0^2}{L^2}\right). \tag{49}$$

For an isotropic beam with $E = 2(1+\nu)G$ and with aspect ratio $L/2r_0 = 5$ the term inside the parentheses in (49) is given approximately by $1-(1+\nu)/30$.

Remark

While the special case $\beta = 0$ of the above result is readily deduced directly on the basis of (34) without the second term on the left we note that within the framework of the analysis in Reissner and Tsai (1972) the center of shear comes out to be coincident with a center of *twist* for which analytical and numerical results *for the limiting case* $\beta = 0$, can be found in Tsai (1972) for the entire class of uniform isotropic circular ring sector cross-section beams.

THE ADDITIONAL DEFLECTION DUE TO TRANSVERSE SHEAR FOR A UNIFORM CIRCULAR CROSS-SECTION BEAM

We now stipulate $V = 0$ and $\alpha = \pi$, with $C = 0$ on the basis of (35), and with g_1 in (39) the same as f_0 in (43). With this we have now

$$\psi = GU\frac{r_0}{L}\left(\frac{r}{r_0}-\frac{I_1(\beta r/r_0)}{\beta I_1'(\beta)}\right)\cos\theta. \tag{50}$$

The introduction of (50) into (36) results in the force–deflection relation

$$\frac{P}{U} = \pi E\frac{r_0^4}{L^3}\int_0^1\left[\rho^3-\left(\rho-\frac{I_1(\beta\rho)}{\beta I_1'(\beta)}\right)\rho^2\right]\mathrm{d}\rho = \pi E\frac{r_0^4}{L^3}\,\frac{I_2(\beta)}{\beta^2 I_1'(\beta)}. \tag{51}$$

With a bending stiffness factor $B = \pi E r_0^4/4$, with $I_1'(\beta)$ in accordance with (48), and with $I_2(\beta) = \beta^2/8+\beta^4/96+\cdots$ (51) gives an approximation for the deflection of the loaded end of the beam:

$$U = \frac{PL^3}{3B}\,\frac{1+3\beta^2/8+\cdots}{1+\beta^2/12+\cdots} \approx \frac{PL^3}{3B}\left(1-\frac{7}{8}\,\frac{E}{G}\,\frac{r_0^2}{L^2}\right). \tag{52}$$

The shear correction term in (52) may be compared with a corresponding term $(6E/5G)(r_0/L)^2$ for the plane stress problem of a narrow rectangular cross-section beam, as discussed in Nair and Reissner (1975).

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